

# Regularity and Cohomological Splitting Conditions for Vector Bundles on Multiprojective Spaces

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February 28, 2008

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## Abstract

Here we give a definition of regularity on multiprojective spaces which is different from the definitions of Hoffmann-Wang and Costa-Miró Roig. By using this notion we prove some splitting criteria for vector bundles.

## 1 Introduction

In chapter 14 of [13] Mumford introduced the concept of regularity for a coherent sheaf on a projective space  $\mathbf{P}^n$ . It was soon clear that Mumford's definition of Castelnuovo-Mumford regularity was a key notion and a fundamental tool in many areas of algebraic geometry and commutative algebra. Several extensions of this notion were proposed to handle different situations ([2], [3], [4], [5] and [8]).

In [2] we introduced the notion of Qregularity on a quadric hypersurface, in order to prove an extension of Evans-Griffiths criterion to vector bundles on Quadrics. In particular we got a new and simple proof of the Knörrer's characterization of ACM bundles.

In this paper we use similar techniques on multiprojective spaces.

Hoffmann and Wang gave the following definition of regularity on  $\mathbf{P}^n \times \mathbf{P}^m$ : a coherent sheaf  $F$  on  $\mathbf{P}^n \times \mathbf{P}^m$  is said to be  $(p, p')$ -regular if, for all  $i > 0$ ,

$$H^i(F(p, p') \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k = -i - 1$ ,  $j < 0$  and  $k < 0$ .

For a definition of regularity on multiprojective spaces (and much more), see [4]. Here we

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<sup>1</sup>Mathematics Subject Classification 2000: 14F05, 14J60.

keywords: vector bundles; multiprojective spaces; Castelnuovo-Mumford regularity.

introduce the following modification of Hoffman and Wang definition on arbitrary multiprojective spaces:

a coherent sheaf  $F$  on  $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$  is said to be  $(p_1, \dots, p_s)$ -regular if, for all  $i > 0$ ,

$$H^i(F(p_1, \dots, p_s) \otimes \mathcal{O}(k_1, \dots, k_s)) = 0$$

whenever  $k_1 + \dots + k_s = -i$  and  $-n_j \leq k_j \leq 0$  for any  $j = 1, \dots, s$ .

In the case  $s = 2$  we have just the following definition:

a coherent sheaf  $F$  on  $\mathbf{P}^n \times \mathbf{P}^m$  is said  $(p, p')$ -regular if, for all  $i > 0$ ,

$$H^i(F(p, p') \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ .

We define  $Reg(F)$  as the least integer  $p$  such that  $F$  is  $(p, p)$ -regular.

In the next section we will prove that our definition of regularity for biprojective spaces (the case  $s = 2$ ) satisfies the analogs of the classical properties on  $\mathbb{P}^n$ . Moreover it has several nice features and allows us to classify some "extremal cases".

In the third section we will apply our definition of regularity in order to prove a few splitting criteria for vector bundles on  $\mathbf{P}^n \times \mathbf{P}^m$ .

A well known result by Horrocks (see [9]) characterizes the vector bundles without intermediate cohomology on a projective space as direct sum of line bundles. This criterion fails on more general varieties. In fact there exist non-split vector bundles without intermediate cohomology. These bundles are called ACM bundles.

On  $\mathbf{P}^n$  all the line bundles are ACM but on  $\mathbf{P}^n \times \mathbf{P}^m$  there are line bundles which are not ACM.

We prove the following extension of the Horrocks criterion on  $\mathbf{P}^n \times \mathbf{P}^m$ :

**Theorem 1.1.** *Let  $E$  be a rank  $r$  vector bundle on  $\mathbf{P}^n \times \mathbf{P}^m$ .*

*Then the following conditions are equivalent:*

1. *for any  $i = 1, \dots, m + n - 1$  and for any integer  $t$ ,*

$$H^i(E(t, t) \otimes \mathcal{O}(j, k)) = 0$$

*whenever  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ .*

2. *There are  $r$  integer  $t_1, \dots, t_r$  such that  $E \cong \bigoplus_{i=1}^r \mathcal{O}(t_i, t_i)$ .*

We prove also the following result:

**Theorem 1.2.** *Let  $E$  be a vector bundle on  $\mathbf{P}^n \times \mathbf{P}^m$ .*

*Then the following conditions are equivalent:*

1. *for any  $i = 1, \dots, m + n - 1$  and for any integer  $t$ ,*

$$H^i(E(t, t) \otimes \mathcal{O}(j, k)) = 0$$

*whenever  $-i \leq j + k \leq 0$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$  but  $(j, k) \neq (-n, 0), (0, -m)$ .*

2.  $E$  is a direct sum of line bundles  $\mathcal{O}$ ,  $\mathcal{O}(0, 1)$  and  $\mathcal{O}(1, 0)$  with some balanced twist  $(t, t)$ .

Theorem 1.2 is the extension to the case  $n, m$  arbitrary of the classification of the ACM bundles on  $\mathcal{Q}_2$  (where  $m = n = 1$ ) proved in [10].

On  $\mathbb{P}^n$ , Evans and Griffith (see [6]) have improved Horrocks' criterion.

We prove also an extension of Evans-Griffiths criterion on  $\mathbf{P}^n \times \mathbf{P}^m$  (see Corollary 3.7).

For a rank  $r$  ( $r < m + n$ ) vector bundle  $E$  we ask the vanishing in (1) of the above theorems only for  $i = 1, \dots, r - 1$  and we add some extra cohomological vanishing condition in order to show that  $E$  splits. This extra conditions do not appear in the Evans-Griffiths criterion on  $\mathbb{P}^n$ . In Theorem 3.13 we show that on  $\mathbf{P}^n \times \mathbf{P}^m$  all these extra hypothesis are necessary and every condition correspond to a direct summand  $\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a + 1)$  (where  $1 \leq a \leq m - 1$ ) or  $\Omega_{\mathbf{P}^n}^a(a + 1) \boxtimes \mathcal{O}$  (where  $1 \leq a \leq n - 1$ ).

We finally specialize on rank two bundles giving the following statement:

**Proposition 1.3.** *Let  $n, m > 2$ . Let  $E$  be a rank 2 vector bundle on  $\mathbf{P}^n \times \mathbf{P}^m$  with  $\text{Reg}(E) = 0$ .*

*Then the following conditions are equivalent:*

1.  $H^1(E(0, 0)) = H^1(E(-1, 0)) = H^1(E(0, -1)) = 0$
2.  $E \cong \mathcal{O} \oplus \mathcal{O}(a, b)$  or  $E \cong \mathcal{O}(0, 1) \oplus \mathcal{O}(a, b)$  or  $E \cong \mathcal{O}(1, 0) \oplus \mathcal{O}(a, b)$  where  $a, b \geq 0$ .

In the last section we generalize our main results to an arbitrary multiprojective space.

We think that our definition is interesting and useful, because our results are of the type "if and only if". Nevertheless, we also feel that the definition of  $(p, p')$ -regularity given in [8] is interesting and useful (as any reader of [8] may see).

We work over an algebraically closed field with characteristic zero. We only need the characteristic zero assumption to prove Theorem 3.13, Corollary 3.7, Corollary 3.16, Proposition 1.3, Theorem 4.8 and Proposition 4.9, because in their proofs we will use Le Potier vanishing theorem.

We thanks E. Arrondo for helpful discussions.

## 2 Regularity on $\mathbf{P}^n \times \mathbf{P}^m$

Let us consider  $X = \mathbf{P}^n \times \mathbf{P}^m$ . We recall the multigraded variant of the Castelnuovo-Mumford regularity introduced by Hoffmann and Wang (see [8]):

**Definition 2.1** (Hoffmann and Wang). *A coherent sheaf  $F$  on  $X$  is said to be  $(p, p')$ -regular if, for all  $i > 0$ ,*

$$H^i(F(p, p') \otimes \mathcal{O}(j, k)) = 0$$

*whenever  $j + k = -i - 1$ ,  $j < 0$  and  $k < 0$ .*

*We will say regular in order to  $(0, 0)$ -regular.*

*We will say  $p$ -regular in order to  $(p, p)$ -regular.*

*We define the HW-regularity of  $F$ ,  $\text{HW} - \text{Reg}(F)$ , as the least integer  $p$  such that  $F$  is  $p$ -regular. We set  $\text{HW} - \text{Reg}(F) = -\infty$  if there is no such integer.*

We give a definition of regularity on  $X$  which is slightly different from the one by Hoffmann and Wang:

**Definition 2.2.** *A coherent sheaf  $F$  on  $X$  is said to be  $(p, p')$ -regular if, for all  $i > 0$ ,*

$$H^i(F(p, p') \otimes \mathcal{O}(j, k)) = 0$$

*whenever  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ .*

*We will say regular in order to  $(0, 0)$ -regular.*

*We will say  $p$ -regular in order to  $(p, p)$ -regular.*

*We define the regularity of  $F$ ,  $\text{Reg}(F)$ , as the least integer  $p$  such that  $F$  is  $p$ -regular. We set  $\text{Reg}(F) = -\infty$  if there is no such integer.*

**Remark 2.3.** *If  $n = 0$  we can identify  $X$  with  $\mathbf{P}^m$  and the sheaf  $F(k, k')$  with  $F(k')$ . Under this identification  $F$  is  $(p, p')$ -regular in the sense of Definition 2.2, if and only if  $F$  is  $p'$ -regular in the sense of Castelnuovo-Mumford.*

*In fact, let  $i > 0$ ,  $H^i(F(p, p') \otimes \mathcal{O}(j, k)) = H^i(F(p' + k)) = 0$  whenever  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$  if and only if  $H^i(F(p' + k)) = 0$  whenever  $-i \leq k \leq 0$  if and only if  $H^i(F(p' - i)) = 0$ .*

**Lemma 2.4.** *Let  $H$  be a generic hyperplane of  $\mathbf{P}^n$ . If  $F$  is a regular coherent sheaf on  $X$ , then  $F|_{L_1}$  is regular on  $L_1 = H \times \mathbf{P}^m$ .*

*The similar statement is true for a generic hyperplane of  $\mathbf{P}^m$ .*

*Proof.* We follow the proof of [8] Lemma 2.6.. We get this exact cohomology sequence:

$$\cdots \rightarrow H^i(F(j, k)) \rightarrow H^i(F|_{L_1}(j, k)) \rightarrow H^{i+1}(F(j-1, k)) \rightarrow \cdots$$

If  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ , we have also  $-n-1 \leq j-1 \leq 0$ , so the first and the third groups vanish by hypothesis. Then also the middle group vanishes and  $F|_{L_1}$  is regular.  $\square$

**Proposition 2.5.** *Let  $F$  be a regular coherent sheaf on  $X$  then*

1.  *$F(p, p')$  is regular for  $p, p' \geq 0$ .*

2.  *$H^0(F(k, k'))$  is spanned by*

$$H^0(F(k-1, k')) \otimes H^0(\mathcal{O}(1, 0))$$

*if  $k-1, k' \geq 0$ ; and it is spanned by*

$$H^0(F(k, k'-1)) \otimes H^0(\mathcal{O}(0, 1))$$

*if  $k, k'-1 \geq 0$ .*

*Proof.* (1) We will prove part (1) by induction. Let  $F$  be a regular coherent sheaf, we want show that also  $F(1, 0)$  is regular. We follow the proof of [8] Proposition 2.7.

Consider the exact cohomology sequence:

$$\cdots \rightarrow H^i(F(j, k)) \rightarrow H^i(F(j+1, k)) \rightarrow H^i(F|_{L_1}(j+1, k)) \rightarrow \cdots$$

If  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ , so the first and the third groups vanish by hypothesis. Then also the middle group vanishes.

A symmetric argument shows the vanishing for  $F(0, 1)$ .

(2) We will follow the proof of [8] Proposition 2.8.

We consider the following diagram:

$$\begin{array}{ccc} H^0(F(k-1, k')) \otimes H^0(\mathcal{O}(1, 0)) & \xrightarrow{\sigma} & H^0(F|_{L_1}(k-1, k')) \otimes H^0(\mathcal{O}_{L_1}(1, 0)) \\ \downarrow \mu & & \downarrow \tau \\ H^0(F(k, k')) & \xrightarrow{\nu} & H^0(F|_{L_1}(k, k')) \end{array}$$

Note that  $\sigma$  is surjective if  $k-1, k' \geq 0$  because  $H^1(F(k-2, k')) = 0$  by regularity.

Moreover also  $\tau$  is surjective by (2) for  $F|_{L_1}$ .

Since both  $\sigma$  and  $\tau$  are surjective, we can see as in [13] page 100 that  $\mu$  is also surjective.  $\square$

**Remark 2.6.** *If  $F$  is a regular coherent sheaf on  $X$  then it is globally generated.*

*In fact by the above proposition we have the following surjections:*

$$H^0(F) \otimes H^0(\mathcal{O}(1, 0)) \otimes H^0(\mathcal{O}(0, 1)) \rightarrow H^0(F(1, 0)) \otimes H^0(\mathcal{O}(0, 1)) \rightarrow H^0(F(1, 1)),$$

and so the map

$$H^0(F) \otimes H^0(\mathcal{O}(1, 1)) \rightarrow H^0(F(1, 1))$$

is a surjection.

Moreover we can consider a sufficiently large twist  $l$  such that  $F(l, l)$  is globally generated.

The commutativity of the diagram

$$\begin{array}{ccc} H^0(F) \otimes H^0(\mathcal{O}(l, l)) \otimes \mathcal{O} & \rightarrow & H^0(F(l, l)) \otimes \mathcal{O} \\ \downarrow & & \downarrow \\ H^0(F) \otimes \mathcal{O}(l, l) & \rightarrow & F(l, l) \end{array}$$

yields the surjectivity of  $H^0(F) \otimes \mathcal{O}(l, l) \rightarrow F(l, l)$ , which implies that  $F$  is generated by its sections.

**Remark 2.7.** *Künneth formula gives that  $\mathcal{O}(a, b)$  is regular if and only if  $a \geq 0$  and  $b \geq 0$ .*

*In fact*

$$H^{n+m}(\mathcal{O}(a-n, b-m)) \cong H^n(\mathcal{O}(a-n)) \otimes H^m(\mathcal{O}(b-m)) = 0$$

*if and only if  $a \geq 0$  or  $b \geq 0$ .*

*Let assume that  $a < 0$  and  $b \geq 0$ , we have*

$$H^n(\mathcal{O}(a-n, b)) \cong H^n(\mathcal{O}(a-n)) \otimes H^0(\mathcal{O}(b)) \neq 0.$$

*This means that if  $\mathcal{O}(a, b)$  is regular we must have  $a \geq 0$  and  $b \geq 0$ .*

*In particular  $\mathcal{O}$  is regular but  $\mathcal{O}(-1, -1)$  is not and so  $\text{Reg}(\mathcal{O}) = 0$ .*

*Moreover in a similar way we can see that  $\text{Reg}(\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1)) = 0$  for any  $1 \leq a \leq m-1$*

Now we want compare the two definitions of regularity.

**Proposition 2.8.** *Let  $F$  be a coherent sheaf on  $X$ , then*

1. *If  $F$  is regular according to Hoffmann and Wang, then it is regular according to Definition 2.2.*

2. If  $F$  is  $(-m+1, -n+1)$ -regular according to Definition 2.2, then it is regular according to Hoffmann and Wang.

*Proof.* (1) Let  $F$  be regular according to Hoffmann and Wang: for all  $i > 0$ ,

$$H^i(F \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k = -i - 1$ ,  $j < 0$  and  $k < 0$ .

By [8] Proposition 2.7.,  $F(p, p')$  is also regular according to Hoffmann and Wang for any  $p \geq 0$  and  $p' \geq 0$ .

In particular we have that for all  $i > 0$ ,

$$H^i(F \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ .

(2) Let  $F$  be  $(-m+1, -n+1)$ -regular according to Definition 2.2. We have

$$H^{m+n}(F(-n-m+1, -m-n+1)) = 0,$$

and so

$$H^{m+n}(F(-n-m+1, -1)) = H^{m+n}(F(-n-m+2, -2)) = \dots = H^{m+n}(F(-1, -m-n+1)) = 0.$$

In the same way all the others vanishing conditions in the definition of regularity by Hoffmann and Wang are satisfied.  $\square$

**Remark 2.9.** Costa and Miró-Roig give a notion of regularity for any  $d$ -dimensional smooth projective variety with a  $d$ -block collection  $\mathcal{B}$  (see [4]).

On  $X$  let us consider the  $(m+n)$ -block collection

$$(\mathcal{E}_0, \dots, \mathcal{E}_{m+n})$$

where for any  $0 \leq j \leq m+n$ , denote by  $\mathcal{E}_j$  the collection of all line bundles on  $X$   $\mathcal{O}(a, b)$  with  $-n \leq a \leq 0$ ,  $-m \leq b \leq 0$  and  $a + b = j - m - n$  (see [4] Example 3.7.(2)).

By [4] Theorem 5.5. a coherent sheaf is regular according to Hoffmann and Wang if and only if it is  $(-n-m)$ -regular with respect to  $\mathcal{B}$ .

So we conclude from the above theorem that a coherent sheaf has regularity  $-\infty$  according to Definition 2.2 if and only if it has regularity  $-\infty$  with respect to  $\mathcal{B}$ .

By [1] Theorem 1 we have

$$\text{Reg}(F) = -\infty \Leftrightarrow \text{Supp}(F) \text{ is finite}$$

### 3 Splitting Criteria on $\mathbf{P}^n \times \mathbf{P}^m$

We use our notion of regularity in order to proving some splitting criterion on  $X = \mathbf{P}^n \times \mathbf{P}^m$ . We need the following definition:

**Definition 3.1.** We say that a vector bundle  $E$  on  $X$  is ACM if for any  $i = 1, \dots, m+n-1$  and for any integer  $t$ ,

$$H^i(E(t, t)) = 0.$$

**Remark 3.2.** *Künneth formula gives that  $\mathcal{O}(a, b)$  is ACM if and only if  $a - b \geq -m$  and  $b - a \geq -n$ .*

*In fact*

$$H^n(\mathcal{O}(a+t, b+t)) \cong H^n(\mathcal{O}(a+t)) \otimes H^0(\mathcal{O}(b+t)) = 0$$

*for any integer  $t$ , if and only if  $b - a \geq -n$ .*

*Moreover*

$$H^m(\mathcal{O}(a+t, b+t)) \cong H^0(\mathcal{O}(a+t)) \otimes H^m(\mathcal{O}(b+t)) = 0$$

*for any integer  $t$ , if and only if  $a - b \geq -m$ .*

*All the others vanishing are satisfied.*

Now are ready to prove Theorem 1.1:

*Proof of Theorem 1.1.* (1)  $\Rightarrow$  (2). Let assume that  $t$  is an integer such that  $E(t, t)$  is regular but  $E(t-1, t-1)$  not.

By the definition of regularity and (1) we can say that  $E(t-1, t-1)$  is not regular if and only if  $H^{m+n}(E(t-1, t-1) \otimes \mathcal{O}(-n, -m)) \neq 0$ . By Serre duality we have that  $H^0(E^\vee(-t, -t)) \neq 0$ . Now since  $E(t, t)$  is globally generated by Remark 2.6 and  $H^0(E^\vee(-t, -t)) \neq 0$  we can conclude that  $\mathcal{O}$  is a direct summand of  $E(t, t)$ .

By iterating these arguments we get (2).

(2)  $\Rightarrow$  (1).  $\mathcal{O}(j, k)$  is ACM whenever,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ . So if  $E \cong \bigoplus_{i=1}^r \mathcal{O}(t_i, t_i)$  then it satisfies all the conditions in (1).

**Remark 3.3.** *If  $n = 0$  the above theorem is the Horrocks criterion on  $\mathbf{P}^m$  (see [9]).*

Now, by adding some cohomological condition and leaving the hypothesis

$$H^n(E(t, t) \otimes \mathcal{O}(-n, 0)) = H^m(F(t, t) \otimes \mathcal{O}(0, -m)) = 0$$

for any integer  $t$ , we are able to prove Theorem 1.2:

*Proof of Theorem 1.2.* (1)  $\Rightarrow$  (2). By Serre duality

$$H^i(E(t, t) \otimes \mathcal{O}(j, k)) \cong H^{m+n-i}(E^\vee(t, t) \otimes \mathcal{O}(-n-1-j, -m-1-k)).$$

Let assume that  $t$  is an integer such that  $E(t, t)$  is regular but  $E(t-1, t-1)$  not.

By the definition of regularity and (1) we can say that  $E(t-1, t-1)$  is not regular if and only if one of the following conditions is satisfied:

- i  $H^{m+n}(E(t-1, t-1) \otimes \mathcal{O}(-n, -m)) \neq 0$ ,
- ii  $H^n(E(t-1, t-1) \otimes \mathcal{O}(-n, 0)) \neq 0$ .
- iii  $H^m(F(t-1, t-1) \otimes \mathcal{O}(0, -m)) \neq 0$ .

Let us consider one by one the conditions:

(i) Let  $H^{m+n}(E(t-1, t-1) \otimes \mathcal{O}(-n, -m)) \neq 0$ , we can conclude that  $\mathcal{O}(t, t)$  is a direct summand as in the above theorem.

(ii) Let  $H^n(E(t-1, t-1) \otimes \mathcal{O}(-n, 0)) \neq 0$ . Let us consider the Koszul complex:

$$0 \rightarrow \mathcal{O}(-n-1, -1) \otimes E(t, t) \rightarrow \mathcal{O}(-n, -1)^{\binom{n+1}{n}} \otimes E(t, t) \rightarrow \dots$$

$$\cdots \rightarrow \mathcal{O}(-1, -1)^{\binom{n+1}{1}} \otimes E(t, t) \rightarrow \mathcal{O}(0, -1) \otimes E(t, t) \rightarrow 0.$$

Since

$$H^n(E(t - n, t - 1)) = \cdots = H^1(E(t - 1, t - 1)) = 0,$$

(if  $n > m$  we use also the hypothesis  $H^m(E(t - m, t - 1)) = 0$ ) we have a surjective map

$$H^0(E(t, t - 1)) \rightarrow H^n(E(t - n - 1, t - 1)).$$

Therefore  $H^0(E(t, t) \otimes \mathcal{O}(0, -1)) \neq 0$  and there exists a non zero map

$$f : E(t, t) \rightarrow \mathcal{O}(0, 1).$$

On the other hand

$$H^n(E(t - n - 1, t - 1)) \cong H^m(E^\vee(-t, -t - m))$$

so let us consider the Koszul complex

$$\begin{aligned} 0 \rightarrow \mathcal{O}(0, -m) \otimes E^\vee(-t, -t) \rightarrow \mathcal{O}(0, -m + 1)^{\binom{m+1}{m}} \otimes E^\vee(-t, -t) \rightarrow \cdots \\ \cdots \rightarrow \mathcal{O}(0, 0)^{\binom{m+1}{1}} \otimes E^\vee(-t, -t) \rightarrow \mathcal{O}(0, 1) \otimes E^\vee(-t, -t) \rightarrow 0. \end{aligned}$$

Since

$$H^n(E(t - n - 1, t - 2)) = \cdots = H^{m+n-1}(E(t - n - 1, t - m - 1)) = 0,$$

(if  $m > n$  we use also the hypothesis  $H^m(E(t - n - 1, t - 2 - m + n)) = 0$ ) and by Serre duality

$$H^m(E^\vee(-t, -t - m + 1)) = \cdots = H^1(E^\vee(-t, -t)) = 0,$$

we have a surjective map

$$H^0(E^\vee(-t, -t + 1)) \rightarrow H^m(E(-t, -t - m)).$$

Therefore  $H^0(E^\vee(-t, -t) \otimes \mathcal{O}(0, 1)) \neq 0$  and there exists a non zero map

$$g : \mathcal{O}(0, 1) \rightarrow E(t, t).$$

Let us consider the following commutative diagram:

$$\begin{array}{ccc} H^n(E(t - n - 1, t - 1)) \otimes H^m(E^\vee(-t, -t - m)) & \xrightarrow{\sigma} & H^{m+n}(\mathcal{O}(-n - 1, -1) \otimes \mathcal{O}(0, -m)) \cong \mathbb{C} \\ \downarrow & & \downarrow \\ H^0(E(t, t - 1)) \otimes H^m(E^\vee(-t, -t - m)) & \xrightarrow{\mu} & H^m(\mathcal{O}(0, -1) \otimes \mathcal{O}(0, -m)) \cong \mathbb{C} \\ \downarrow & & \downarrow \\ H^0(E(t, t - 1)) \otimes H^0(E^\vee(-t, -t + 1)) & \xrightarrow{\tau} & H^0(\mathcal{O}(0, -1) \otimes \mathcal{O}(0, 1)) \cong \mathbb{C} \\ \uparrow \cong & & \uparrow \cong \\ \text{Hom}(E(t, t), \mathcal{O}(0, 1)) \otimes \text{Hom}(\mathcal{O}(0, 1), E(t, t)) & \xrightarrow{\gamma} & \text{Hom}(\mathcal{O}(0, 1), \mathcal{O}(0, 1)). \end{array}$$

The map  $\sigma$  comes from Serre duality and it is not zero, the right vertical map are isomorphisms and the left vertical map are surjective so also the map  $\tau$  is not zero.

This means that the the map

$$f \circ g : \mathcal{O}(0, 1) \rightarrow \mathcal{O}(0, 1)$$

is non-zero and hence it is an isomorphism.

This isomorphism shows that  $\mathcal{O}(0, 1)$  is a direct summand of  $E(t, t)$ .

(iii) let  $H^m(F(t - 1, t - 1) \otimes \mathcal{O}(0, -m)) \neq 0$ . By arguing as above we can conclude that  $\mathcal{O}(1, 0)$  is a direct summand of  $E(t, t)$ .

(2)  $\Rightarrow$  (1). As in Theorem 1.1. □



**Remark 3.4.** If  $n = m = 1$  we have exactly the classification of the ACM bundles on  $\mathcal{Q}_2$  (see [10]).

The proof in this case coincides with [2] Theorem 1.2.

**Remark 3.5.** Let  $E$  be a vector bundle on  $X$ . Let  $a$  and  $b$  be two integers. Then the following conditions are equivalent:

1. for any  $i = 1, \dots, m + n - 1$  and for any integer  $t$ ,

$$H^i(E(a + t, b + t) \otimes \mathcal{O}(j, k)) = 0$$

whenever  $-i \leq j + k \leq 0$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$  but  $(j, k) \neq (-n, 0), (0, -m)$ .

2.  $E$  is a direct sum of line bundles  $\mathcal{O}(a, b)$ ,  $\mathcal{O}(a, b + 1)$  and  $\mathcal{O}(a + 1, b)$  with some balanced twist  $(t, t)$ .

*Proof.* Let us consider  $E \otimes \mathcal{O}(-a, -b)$  and let us apply the above theorem. □

**Remark 3.6.** If we add in the conditions (1) of the above remark the hypothesis

$$H^{m+n}(E(t + a, t + b) \otimes \mathcal{O}(-n, -m)) = 0$$

for any integer  $t$ , we conclude that  $E$  can be only a direct sum of line bundles  $\mathcal{O}(a, b + 1)$  and  $\mathcal{O}(a + 1, b)$  with some balanced twist  $(t, t)$ .

If we add in the conditions (1) of the above remark the hypothesis

$$H^n(E(t + a, t + b) \otimes \mathcal{O}(-n, 0)) = 0$$

for any integer  $t$ , we conclude that  $E$  can be only a direct sum of line bundles  $\mathcal{O}(a, b)$  and  $\mathcal{O}(a, b + 1)$  with some balanced twist  $(t, t)$ .

By applying Le Potier vanishing Theorem we can prove the following extension of Evans-Griffiths criterion to vector bundles on  $X$ :

**Corollary 3.7.** Let  $E$  be a rank  $r$  ( $r < n + m$ ) vector bundle on  $X$ . Then the following conditions are equivalent:

1. for any  $i = 1, \dots, r - 1$  and for any integer  $t$ ,

$$H^i(E(t, t) \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k \geq -i$ ,  $-n < j \leq 0$  and  $-m < k \leq 0$ .

Moreover for any  $i = 1, \dots, m + n - 1$  but  $i \neq n, m$  and for any integer  $t$ ,

$$H^i(F(t, t) \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k = -i$ , and  $j = -n$  or  $k = -m$ .

2.  $E$  is a direct sum of line bundles  $\mathcal{O}$ ,  $\mathcal{O}(0, 1)$  and  $\mathcal{O}(1, 0)$  with some balanced twist  $(t, t)$ .

*Proof.* (1)  $\Rightarrow$  (2). Let assume that  $t$  is an integer such that  $E(t, t)$  is regular but  $E(t-1, t-1)$  not.

$E(t, t)$  is globally generated by Remark 2.6. Since the tensor product of a spanned vector bundle by an ample vector bundle is ample (see [7] Corollary III.1.9), we have

$$a, b > t \Rightarrow E(a, b) \text{ is ample .}$$

So, by Le Potier vanishing theorem, we have that  $H^i(E^\vee(-a, -b)) = 0$  for every  $a, b > t$  and  $i = 1, \dots, n + m - r$ .

So by Serre duality  $H^i(E(-n-1+a, -m-1+b)) = 0$  for every  $a, b > t$  and  $i = r, \dots, n+m-1$ . By the definition of regularity, this vanishing and (1) we can say that  $E(t-1, t-1)$  is not regular if and only if  $H^{m+n}(E(t-1, t-1) \otimes \mathcal{O}(-n, -m)) \neq 0$ . We can conclude that  $\mathcal{O}$  is a direct summand of  $E(t, t)$  as in the above theorem.

By iterating these arguments we get (2).

(2)  $\Rightarrow$  (1). See the above theorem.  $\square$

**Remark 3.8.** *If we add in (1) the condition*

$$H^i(E(t, t) \otimes \mathcal{O}(-n, 0)) = H^i(E(t, t) \otimes \mathcal{O}(0, -m)) = 0$$

*for any integer  $t$ , we can conclude that there are  $r$  integer  $t_1, \dots, t_r$  such that  $E \cong \bigoplus_{i=1}^r \mathcal{O}(t_i, t_i)$ .*

**Remark 3.9.** *The following conditions:*

*for any integer  $t$ , and  $i = 1, \dots, m + n - 1$ ,*

$$H^i(F(t, t) \otimes \mathcal{O}(j, k)) = 0$$

*whenever  $j + k = -i$ , and  $j = -n$  or  $k = -m$  do not appear in the Evans-Griffiths criterion on  $\mathbb{P}^n$ . On  $X$  they are necessary.*

*In fact we can find many bundles on  $X$  with all the vanishing in (1) except some of the above conditions.*

*Let see some example:*

**Example 3.10.**  $\mathcal{O}(-1, a)$  with  $a \geq 0$ , satisfies all the conditions except  $H^n(\mathcal{O}(-1-n, a)) \neq 0$ .

**Example 3.11.**  $\mathcal{O}(-1) \boxtimes \Omega_{\mathbb{P}^m}^1(1)$  satisfies all the conditions except  $H^n(\mathcal{O}(-1-n) \boxtimes \Omega_{\mathbb{P}^m}^1(1)) \neq 0$  and  $H^{n+1}(\mathcal{O}(-1-n) \boxtimes \Omega_{\mathbb{P}^m}^1(1-1)) \neq 0$ .

**Example 3.12.** More in general  $\mathcal{O}(-1) \boxtimes \Omega_{\mathbb{P}^m}^a(a)$  with  $1 \leq a \leq m-1$ , satisfies all the conditions except

$H^n(\mathcal{O}(-a-n) \boxtimes \Omega_{\mathbb{P}^m}^a(a)) \neq 0$  and  $H^{n+a}(\mathcal{O}(-a-n) \boxtimes \Omega_{\mathbb{P}^m}^a(a-a)) \neq 0$ .

We show now that above examples are the only possible.

**Theorem 3.13.** *Let  $E$  be a rank  $r$  vector bundle on  $\mathbb{P}^n \times \mathbb{P}^m$  with  $\text{Reg}(E) = 0$ .*

*Then the following conditions are equivalent:*

1. *for any  $i = 1, \dots, \min(r, m + n) - 1$ ,*

$$H^i(E(-1, -1) \otimes \mathcal{O}(j, k)) = 0$$

*whenever  $j + k \geq -i$ ,  $-n < j \leq 0$  and  $-m < k \leq 0$ .*

2.  $E$  has one of the following bundles as a direct summand:  $\mathcal{O}$ ,  $\mathcal{O}(0,1)$ ,  $\mathcal{O}(1,0)$ ,  $\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1)$  (where  $1 \leq a \leq m-1$ ) or  $\Omega_{\mathbf{P}^n}^a(a+1) \boxtimes \mathcal{O}$  (where  $1 \leq a \leq n-1$ ).

*Proof.* (1)  $\Rightarrow$  (2). Since  $\text{Reg}(E) = 0$ ,  $E$  is regular but  $E(-1, -1)$  not.

$E$  is globally generated by Remark 2.6. Since the tensor product of a spanned vector bundle by an ample vector bundle is ample (see [7] Corollary III.1.9), we have

$$a, b > 0 \Rightarrow E(a, b) \text{ is ample.}$$

Let assume  $r < m+n$ . So, by Le Potier vanishing theorem, we have that  $H^i(E^\vee(-a, -b)) = 0$  for every  $a, b > 0$  and  $i = 1, \dots, n+m-r$ .

So by Serre duality  $H^i(E(-n-1+a, -m-1+b)) = 0$  for every  $a, b > 0$  and  $i = r, \dots, n+m-1$ . By the definition of regularity, this vanishing and (1) we can say that  $E(-1, -1)$  is not regular if and only if one of the following conditions is satisfied (if  $r \geq m+n$  we can conclude this without using Le Potier vanishing theorem):

- i  $H^{m+n}(E(-1, -1) \otimes \mathcal{O}(-n, -m)) \neq 0$ ,
- ii  $H^n(E(-1, -1) \otimes \mathcal{O}(-n, 0)) \neq 0$ .
- iii  $H^m(E(-1, -1) \otimes \mathcal{O}(0, -m)) \neq 0$ .
- iv There exists an integer  $a$  ( $1 \leq a \leq m-1$ ) such that  $H^{n+a}(E(-1, -1) \otimes \mathcal{O}(-n, -a)) \neq 0$ .
- v There exists an integer  $a$  ( $1 \leq a \leq n-1$ ) such that  $H^{m+a}(E(-1, -1) \otimes \mathcal{O}(-a, -m)) \neq 0$ .

By Theorem 1.2 we know that the conditions [i], [ii] and [iii] give us direct summands  $\mathcal{O}$ ,  $\mathcal{O}(0,1)$  and  $\mathcal{O}(1,0)$ .

Let us consider the others conditions:

[iv] We fix  $a = 1$ . Let  $H^{n+1}(E(-1, -1) \otimes \mathcal{O}(-n, -1)) \neq 0$ . Let us consider the two Koszul complexes:

$$0 \rightarrow \mathcal{O}(-n-1, -2) \rightarrow \mathcal{O}(-n, -2)^{\binom{n+1}{n}} \rightarrow \dots \rightarrow \mathcal{O}(-1, -2)^{\binom{n+1}{1}} \rightarrow \mathcal{O}(0, -2) \rightarrow 0,$$

and the dual of

$$0 \rightarrow \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2) \rightarrow \mathcal{O}(0, 1)^{m+1} \rightarrow \mathcal{O}(0, 2) \rightarrow 0.$$

We tensor by  $E$  and we obtain

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-n-1, -2) \otimes E \rightarrow \mathcal{O}(-n, -2)^{\binom{n+1}{n}} \otimes E \rightarrow \dots \\ \dots \rightarrow \mathcal{O}(-1, -2)^{\binom{n+1}{1}} \otimes E \rightarrow \mathcal{O}(0, -1)^{m+1} \otimes E \rightarrow \mathcal{O} \boxtimes \mathcal{T}_{\mathbf{P}^m}(-2) \otimes E \rightarrow 0, \end{aligned}$$

Since

$$H^{n+1}(E(-n, -1)) = \dots = H^2(E(-1, -2)) = H^1(E(0, -1)) = 0,$$

we have a surjective map

$$H^0(\mathcal{O} \boxtimes \mathcal{T}_{\mathbf{P}^m}(-2) \otimes E) \rightarrow H^n(E(-n-1, -2)).$$

Therefore  $H^0(\mathcal{O} \boxtimes \mathcal{T}_{\mathbf{P}^m}(-2) \otimes E) \neq 0$  and there exists a non zero map

$$f : E \rightarrow \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2).$$

On the other hand

$$H^{n+1}(E(-n-1, -2)) \cong H^{m-1}(E^\vee(0, -m+1))$$

so let us consider the Koszul complex

$$\begin{aligned} 0 \rightarrow \mathcal{O}(0, -m+1) \otimes E^\vee &\rightarrow \mathcal{O}(0, -m+2) \binom{m+1}{m} \otimes E^\vee \rightarrow \dots \\ \dots \rightarrow \mathcal{O}(0, 0) \binom{m+1}{2} \otimes E^\vee &\rightarrow \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2) \otimes E^\vee \rightarrow 0. \end{aligned}$$

Since

$$H^{m-1}(E^\vee(0, -m+2)) = \dots = H^1(E^\vee) = 0,$$

we have a surjective map

$$H^0(\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2) \otimes E^\vee) \rightarrow H^{m-1}(E(-t, -t-m+1)).$$

Therefore  $H^0(\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2) \otimes E^\vee) \neq 0$  and there exists a non zero map

$$g : \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2) \rightarrow E.$$

Now by arguing as in the proof of Theorem 1.2 we can conclude that  $\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2)$  is a direct summand of  $E$ .

In the same way, for any  $a = 1, \dots, m-1$ , we can prove that if

$H^{n+a}(E(-1, -1) \otimes \mathcal{O}(-n, -a)) \neq 0$  then  $\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1)$  is a direct summand of  $E$ .

We need to consider the two Koszul complexes (tensored by  $E$ ):

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-n-1, -a-1) &\rightarrow \mathcal{O}(-n, -a-1) \binom{n+1}{n} \rightarrow \dots \\ \dots \rightarrow \mathcal{O}(-1, -a-1) \binom{n+1}{1} &\rightarrow \mathcal{O}(0, -a-1) \rightarrow 0, \end{aligned}$$

and the dual of

$$\begin{aligned} 0 \rightarrow \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1) &\rightarrow \mathcal{O}(0, 1) \binom{m+1}{a} \rightarrow \mathcal{O}(0, 2) \binom{m+1}{a-1} \dots \\ \dots \rightarrow \mathcal{O}(0, a) \binom{m+1}{1} &\rightarrow \mathcal{O}(0, a+1) \rightarrow 0. \end{aligned}$$

On the other hand

$$H^{n+a}(E(-n-1, -a-1)) \cong H^{m-a}(E^\vee(0, -m+a))$$

so we need to consider the Koszul complex (tensored by  $E^\vee$ )

$$\begin{aligned} 0 \rightarrow \mathcal{O}(0, -m+a) &\rightarrow \mathcal{O}(0, -m+a+1) \binom{m+1}{m} \rightarrow \mathcal{O}(0, -m+a+2) \binom{m+1}{m-1} \rightarrow \dots \\ \dots \rightarrow \mathcal{O}(0, -m+a+1+m-a-1) \binom{m+1}{m-m+a+1} &\rightarrow \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1) \rightarrow 0. \end{aligned}$$

[v] As above.

(2)  $\Rightarrow$  (1). We have to check that for any  $a = 1, \dots, m-1$ ,  $\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1)$  satisfies all the conditions of (1).

Let us consider all the groups of cohomology that can be different from zero:

$H^a(\mathcal{O}(j) \boxtimes \Omega_{\mathbf{P}^m}^a(a+1+k)) \neq 0$  if and only if  $j \geq 0$  and  $k = -a-1$ ,

$H^n(\mathcal{O}(j) \boxtimes \Omega_{\mathbf{P}^m}^a(a+1+k)) \neq 0$  if and only if  $j \leq -n-1$  and  $k \geq -1$ ,

$H^m(\mathcal{O}(j) \boxtimes \Omega_{\mathbf{P}^m}^a(a+1+k)) \neq 0$  if and only if  $j \geq 0$  and  $k \leq -m-a-1$ , and

$H^{n+a}(\mathcal{O}(j) \boxtimes \Omega_{\mathbf{P}^m}^a(a+1+k)) \neq 0$  if and only if  $j \leq -n-1$  and  $k = -a-1$ .

So the conditions (1) are all satisfied.  $\square$

**Remark 3.14.** Fix integers  $n > 0$ ,  $m > 0$  and any  $S \subseteq \{1, \dots, m-1\}$ ,  $S' \subseteq \{1, \dots, n-1\}$ . Make all the assumptions of Theorem 3.13. Add the assumptions

$$H^{n+b}(E(-1, -1) \otimes \mathcal{O}(-n, -b)) = H^{m+c}(E(-1, -1) \otimes \mathcal{O}(-c, -m)) = 0$$

for all  $b \in \{1, \dots, m-1\} - S$  and  $c \in \{1, \dots, n-1\} - S'$ .

The proof of Theorem 3.13 shows that  $E$  has a factor isomorphic to either  $\mathcal{O}$  or  $\mathcal{O}(0, 1)$  or  $\mathcal{O}(1, 0)$  or one of the bundles  $\mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^a(a+1)$  for some  $a \in S$  or one of the bundles  $\Omega_{\mathbf{P}^n}^{a'}(a'+1) \boxtimes \mathcal{O}$  for some  $a' \in S'$ .

**Remark 3.15.** If  $r \geq n$  or  $r \geq m$  Theorem 3.13 cannot become a splitting criterion because it is not possible to iterate the above argument.

If  $r < n, m$  we have the following Corollary:

**Corollary 3.16.** Let  $E$  be a rank  $r$  vector bundle on  $\mathbf{P}^n \times \mathbf{P}^m$ . Let  $r < n, m$ . Then the following conditions are equivalent:

1. for any  $i = 1, \dots, r-1$  and for any integer  $t$ ,

$$H^i(E(t, t) \otimes \mathcal{O}(j, k)) = 0$$

whenever  $j + k \geq -i$ ,  $j, k \leq 0$ .

2.  $E$  is a direct sum of line bundles  $\mathcal{O}$ ,  $\mathcal{O}(0, 1)$  and  $\mathcal{O}(1, 0)$  with some balanced twist  $(t, t)$ .

*Proof.* (1)  $\Rightarrow$  (2). Let assume that  $t$  is an integer such that  $E(t, t)$  is regular but  $E(t-1, t-1)$  not. This means that  $\text{Reg}(E(t, t)) = 0$  so we can apply Theorem 3.13. Since  $r < m, n$ , only the line bundles  $\mathcal{O}$ ,  $\mathcal{O}(0, 1)$  or  $\mathcal{O}(1, 0)$  can be direct summands of  $E(t, t)$ . By iterating this argument we get (2).

(2)  $\Rightarrow$  (1). See Theorem 1.2. □

Now we specialize to the case:  $\text{rank } E = 2$ .

*Proof of Proposition 1.3.* Since  $\text{Reg}(E) = 0$ ,  $E$  is regular but  $E(-1, -1)$  not. By the proof of Theorem 3.13 and by considering that  $\text{rank}(\Omega_{\mathbf{P}^n}^1) > 2$  and  $\text{rank}(\Omega_{\mathbf{P}^m}^1) > 2$ , we have that  $\mathcal{O}$ ,  $\mathcal{O}(0, 1)$ , or  $\mathcal{O}(1, 0)$  is a direct summand of  $E$ .

The other summand must be  $\mathcal{O}(a, b)$  where  $a, b \geq 0$ , by Remark 2.7. □

**Remark 3.17.** If  $m \leq 2$  we have to add to (2) of the above Proposition the possibility  $E \cong \mathcal{O} \boxtimes \Omega_{\mathbf{P}^m}^1(2)$ .

## 4 Generalization on $\mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_s}$

We can easily generalize the notion of regularity on  $X = \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_s}$  ( $d = n_1 + \dots + n_s$ ):

**Definition 4.1.** A coherent sheaf  $F$  on  $X = \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_s}$  is said to be  $(p_1, \dots, p_s)$ -regular if, for all  $i > 0$ ,

$$H^i(F(p_1, \dots, p_s) \otimes \mathcal{O}(k_1, \dots, k_s)) = 0$$

whenever  $k_1 + \dots + k_s = -i$  and  $-n_j \leq k_j \leq 0$  for any  $j = 1, \dots, s$ .

**Remark 4.2.** *Künneth formula gives that  $\mathcal{O}(a_1, \dots, a_s)$  is ACM if and only if for any  $j = 1, \dots, s$  there are  $h, k \neq j$  such that  $a_j - a_h \leq n_h$  and  $a_j - a_k \geq -n_j$ .  
In fact for any  $j = 1, \dots, s$*

$$H^{n_j}(\mathcal{O}(a_1+t, \dots, a_j+t, \dots, a_s+t)) \cong H^{n_j}(\mathcal{O}(a_j+t)) \otimes H^0(\mathcal{O}(a_1+t)) \otimes \dots \otimes H^0(\mathcal{O}(a_s+t)) = 0$$

for any integer  $t$ , if and only if  $a_j - a_k \geq -n_j$  for some  $k \neq j$ .

Moreover

$$H^{d-n_j}(\mathcal{O}(a_1+t, \dots, a_j+t, \dots, a_s+t)) \cong H^0(\mathcal{O}(a_j+t)) \otimes H^{n_1}(\mathcal{O}(a_1+t)) \otimes \dots \otimes H^{n_s}(\mathcal{O}(a_s+t)) = 0$$

for any integer  $t$ , if and only if  $a_h - a_j \geq -n_h$  for some  $h \neq j$ .

All the others vanishing are satisfied.

**Lemma 4.3.** *Let  $H$  be a generic hyperplane of  $\mathbf{P}^{n_1}$ . If  $F$  is a regular coherent sheaf on  $X$ , then  $F|_{L_1}$  is regular on  $L_1 = H \times \mathbf{P}^{n_2} \times \dots \times \mathbf{P}^{n_s}$ .*

*The similar statement is true for a generic hyperplane of any  $\mathbf{P}^{n_j}$ .*

*Proof.* We follow the proof of [8] Lemma 2.6.. We get this exact cohomology sequence:

$$\dots \rightarrow H^i(F(k_1, \dots, k_s)) \rightarrow H^i(F|_{L_1}(k_1, \dots, k_s)) \rightarrow H^{i+1}(F(k_1-1, \dots, k_s)) \rightarrow \dots$$

If  $k_1 + \dots + k_s = -i$  and  $-n_j \leq k_j \leq 0$  for any  $j = 1, \dots, s$ , we have also  $-n_1 - 1 \leq k_1 - 1 \leq 0$ , so the first and the third groups vanish by hypothesis. Then also the middle group vanishes and  $F|_{L_1}$  is regular.  $\square$

**Proposition 4.4.** *Let  $F$  be a regular coherent sheaf on  $X$ . Then*

1.  *$F(p_1, \dots, p_s)$  is regular for  $p_1, \dots, p_s \geq 0$ .*

2. *For any  $j = 1, \dots, s$ ,  $H^0(F(k_1, \dots, k_s))$  is spanned by*

$$H^0(F(k_1, \dots, k_j - 1, \dots, k_s)) \otimes H^0(\mathcal{O}(0, \dots, 1, \dots, 0))$$

*if  $k_1, \dots, k_j - 1, \dots, k_s \geq 0$ .*

*Proof.* (1) We will prove part (1) by induction. We follow the proof of [8] Proposition 2.7. Consider the exact cohomology sequence:

$$\dots \rightarrow H^i(F(k_1, \dots, k_s)) \rightarrow H^i(F(k_1+1, \dots, k_s)) \rightarrow H^{i+1}(F|_{L_1}(k_1+1, \dots, k_s)) \rightarrow \dots$$

If  $j+k = -i$ ,  $-n \leq j \leq 0$  and  $-m \leq k \leq 0$ , so the first and the third groups vanish by hypothesis. Then also the middle group vanishes.

A symmetric argument shows the vanishing for  $F(0, 1, 0, \dots, 0)$  and so on.

(2) We can follow the proof of [8] Proposition 2.8. since we have  $H^1(F(k-1, k')) = 0$ .

We consider the following diagram:

$$\begin{array}{ccc} H^0(F(k_1-1, \dots, k_s)) \otimes H^0(\mathcal{O}(1, 0, \dots, 0)) & \xrightarrow{\sigma} & H^0(F|_{L_1}(k-1)) \otimes H^0(\mathcal{O}_{L_1}(1, 0, \dots, 0)) \\ \downarrow \mu & & \downarrow \tau \\ H^0(F(k_1, \dots, k_s)) & \xrightarrow{\nu} & H^0(F|_{L_1}(k_1, \dots, k_s)) \end{array}$$

Note that  $\sigma$  is surjective if  $k_1 - 1, \dots, k_s \geq 0$  because  $H^1(F(k_1-2, k_2, \dots, k_s)) = 0$  by regularity.

Moreover also  $\tau$  is surjective by (2) for  $F|_{L_1}$ .

Since both  $\sigma$  and  $\tau$  are surjective we can see as in [13] page 100 that  $\mu$  is also surjective.  $\square$

**Remark 4.5.** *If  $F$  is a regular coherent sheaf on  $X$  then it is globally generated. In fact by the above proposition we have the following surjection:*

$$H^0(F) \otimes H^0(\mathcal{O}(1, \dots, 1)) \rightarrow H^0(F(1, \dots, 1))$$

*Moreover we can consider a sufficiently large twist  $l$  such that  $F(l, \dots, l)$  is globally generated. The commutativity of the diagram*

$$\begin{array}{ccc} H^0(F) \otimes H^0(\mathcal{O}(l, \dots, l)) \otimes \mathcal{O} & \rightarrow & H^0(F(l, \dots, l)) \otimes \mathcal{O} \\ \downarrow & & \downarrow \\ H^0(F) \otimes \mathcal{O}(l, \dots, l) & \rightarrow & F(l, \dots, l) \end{array}$$

*yields the surjectivity of  $H^0(F) \otimes \mathcal{O}(l, \dots, l) \rightarrow F(l, \dots, l)$ , which implies that  $F$  is generated by its sections.*

We can now give the following splitting criterion which is the generalization of Theorem 1.1:

**Theorem 4.6.** *Let  $E$  be a rank  $r$  vector bundle on  $X = \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_s}$  and  $d = n_1 + \dots + n_s$ . Then the following conditions are equivalent:*

1. *for any  $i = 1, \dots, d-1$  and for any integer  $t$ ,  $H^i(F(t, \dots, t) \otimes \mathcal{O}(k_1, \dots, k_s)) = 0$  whenever  $k_1 + \dots + k_s = -i$  and  $-n_j \leq k_j \leq 0$  for any  $j = 1, \dots, s$ .*
2. *There are  $r$  integer  $t_1, \dots, t_r$  such that  $E \cong \bigoplus_{i=1}^r \mathcal{O}(t_i, \dots, t_i)$ .*

*Proof.* (1)  $\Rightarrow$  (2). Let assume that  $t$  is an integer such that  $E(t, \dots, t)$  is regular but  $E(t-1, \dots, t-1)$  not.

By the definition of regularity and (1) we can say that  $E(t-1, \dots, t-1)$  is not regular if and only if  $H^d(E(t-1, \dots, t-1) \otimes \mathcal{O}(-n_1, \dots, -n_s)) \neq 0$ . By Serre duality we have that  $H^0(E^\vee(-t, \dots, -t)) \neq 0$ .

Now since  $E(t, \dots, t)$  is globally generated by Remark 4.5 and  $H^0(E^\vee(-t, \dots, -t)) \neq 0$  we can conclude that  $\mathcal{O}$  is a direct summand of  $E(t, \dots, t)$ .

By iterating these arguments we get (2).

(2)  $\Rightarrow$  (1).  $\mathcal{O}(k_1, \dots, k_s)$  is ACM whenever  $-n_j \leq k_j \leq 0$  for any  $j = 1, \dots, s$ . So if  $E \cong \bigoplus_{i=1}^r \mathcal{O}(t_i, \dots, t_i)$  then it satisfies all the conditions in (1).  $\square$

We can also generalize Theorem 1.2:

**Theorem 4.7.** *Let  $E$  be a rank  $r$  vector bundle on  $X = \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_s}$  and  $d = n_1 + \dots + n_s$ . Then the following conditions are equivalent:*

1. *for any  $i = 1, \dots, d-1$  and for any integer  $t$ ,  $H^i(F(t, \dots, t) \otimes \mathcal{O}(k_1, \dots, k_s)) = 0$  whenever  $k_1 + \dots + k_s \geq -i$  and  $-n_j \leq k_j \leq 0$  for any  $j = 1, \dots, s$  but there is an index  $j$  such that  $k_j \neq 0, -n_j$ .*
2.  *$E$  is a direct sum of line bundles  $\mathcal{O}(l_1, \dots, l_s)$  (where for any  $j = 1, \dots, s$   $l_j = 1$  or  $l_j = 0$ ) with some balanced twist  $(t, \dots, t)$ .*

*Proof.* (1)  $\Rightarrow$  (2). Let assume that  $t$  is an integer such that  $E(t, \dots, t)$  is regular but  $E(t-1, \dots, t-1)$  not.

By the definition of regularity and (1) we can say that  $E(t-1, \dots, t-1)$  is not regular if and only if one of the following conditions is satisfied:

- i  $H^d(E(t-1, \dots, t-1) \otimes \mathcal{O}(-n_1, \dots, -n_s)) \neq 0$ ,
- ii there are  $s$  numbers  $h_1, \dots, h_s$  where for any  $j = 1, \dots, s$   $h_j = 0$  or  $h_j = n_j$  and  $0 < h_1 + \dots + h_s < d$  such that  $H^{h_1 + \dots + h_s}(E(t-1, \dots, t-1) \otimes \mathcal{O}(-h_1, \dots, -h_s)) \neq 0$ .

Let us consider one by one the conditions:

(i) Let  $H^d(E(t-1, \dots, t-1) \otimes \mathcal{O}(-n_1, \dots, -n_s)) \neq 0$ , we can conclude that  $\mathcal{O}(t, t)$  is a direct summand as in the above theorem.

(ii) Up to a permutation of the factors of the multiprojective space we may assume that there is an integer  $l$  with  $1 \leq l < s$  such that

$$H^{n_1 + \dots + n_l}(E(t-1, \dots, t-1) \otimes \mathcal{O}(-n_1, \dots, -n_l, 0, \dots, 0)) \neq 0.$$

Let us consider the following exact sequences tensored by  $E(t, \dots, t)$ :

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-n_1-1, \dots, -n_l-1, -1, \dots, -1) \rightarrow \dots \rightarrow \mathcal{O}(0, n_2-1, \dots, -n_l-1, -1, \dots, -1) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}(0, -n_2-1, \dots, -n_l-1, -1, \dots, -1) \rightarrow \dots \rightarrow \mathcal{O}(0, 0, n_3-1, \dots, -n_l-1, -1, \dots, -1) \rightarrow 0, \\ \dots \\ 0 \rightarrow \mathcal{O}(0, \dots, 0, -n_l-1, -1, \dots, -1) \rightarrow \dots \rightarrow \mathcal{O}(0, \dots, 0, -1, \dots, -1) \rightarrow 0. \end{aligned}$$

By using the vanishing conditions in (1) we can conclude that

$$H^0(E(t, \dots, t) \otimes \mathcal{O}(0, \dots, 0, -1, \dots, -1)) \neq 0.$$

On the other hand

$$\begin{aligned} H^{n_1 + \dots + n_l}(E(t-1, \dots, t-1) \otimes \mathcal{O}(-n_1, \dots, -n_l, 0, \dots, 0)) &\cong \\ &\cong H^{n_{l+1} + \dots + n_s}(E^\vee(-t, \dots, -t) \otimes (0, \dots, 0, -n_{l+1}, \dots, -n_s)). \end{aligned}$$

Let us consider the following exact sequences tensored by  $E^\vee(-t, \dots, -t)$ :

$$\begin{aligned} 0 \rightarrow \mathcal{O}(0, \dots, 0, -n_{l+1}, \dots, -n_s) \rightarrow \dots \rightarrow \mathcal{O}(0, \dots, 0, 1, -n_{l+2}, \dots, -n_s) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}(0, \dots, 0, 1, -n_{l+2}, \dots, -n_s) \rightarrow \dots \rightarrow \mathcal{O}(0, \dots, 0, 1, 1, -n_{l+3}, \dots, -n_s) \rightarrow 0, \\ \dots \\ 0 \rightarrow \mathcal{O}(0, \dots, 0, 1, \dots, 1, -n_s) \rightarrow \dots \rightarrow \mathcal{O}(0, \dots, 0, 1, \dots, 1) \rightarrow 0. \end{aligned}$$

By using the vanishing conditions in (1) we can conclude that

$$H^0(E^\vee(-t, \dots, -t) \otimes \mathcal{O}(0, \dots, 0, 1, \dots, 1)) \neq 0.$$

So by arguing as in Theorem 1.2 we have that  $\mathcal{O}(0, \dots, 0, 1, \dots, 1)$  is a direct summand of  $E(t, \dots, t)$ .  $\square$



We finally give also the generalization of Theorem 3.13:

**Theorem 4.8.** *Let  $E$  be a rank  $r$  vector bundle on  $X = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$  and  $d = n_1 + \cdots + n_s$  with  $\text{Reg}(E) = 0$ .*

*Then the following conditions are equivalent:*

1. *for any  $i = 1, \dots, \min(r, d) - 1$  and for any integer  $t$ ,  $H^i(E(t, \dots, t) \otimes \mathcal{O}(k_1, \dots, k_s)) = 0$  whenever  $k_1 + \dots + k_s \geq -i$  and  $-n_j < k_j \leq 0$  for any  $j = 1, \dots, s$ .*
2.  *$E$  has one of the following bundles as a direct summand:  
 $\mathcal{O}(l_1, \dots, l_s)$  (where for any  $j = 1, \dots, s$ ,  $l_j = 1$  or  $l_j = 0$  but  $(l_1, \dots, l_s) \neq (1, \dots, 1)$ )  
and bundles  $A_{l(1)} \boxtimes \cdots \boxtimes A_{l(s)}$ , (where for any  $j = 1, \dots, s$   $l(j) = 1, \dots, n_j$  and  
 $A_{l(j)} \cong \Omega_{\mathbf{P}^{n_j}}^{l(j)}(l(j) + 1)$ ). Moreover at least one of the  $A_{l(j)}$  must be  $\mathcal{O}$ ).*

*Proof.* (1)  $\Rightarrow$  (2). Since  $\text{Reg}(E) = 0$ ,  $E$  is regular but  $E(-1, \dots, -1)$  not.  $E$  is globally generated by Remark 4.5. Since the tensor product of a spanned vector bundle by an ample vector bundle is ample (see [7] Corollary III.1.9), we have

$$a_1, \dots, a_s > 0 \Rightarrow E(a_1, \dots, a_s) \text{ is ample .}$$

Let assume  $r < d$ . So, by Le Potier vanishing theorem, we have that  $H^i(E^\vee(-a_1, \dots, -a_s)) = 0$  for every  $a_1, \dots, a_s > 0$  and  $i = 1, \dots, d - r$ .

So by Serre duality  $H^i(E(-n_1 - 1 + a_1, \dots, -n_s - 1 + a_s)) = 0$  for every  $a_1, \dots, a_s > 0$  and  $i = r, \dots, d - 1$ .

By the definition of regularity, this vanishing and (1) we can say that  $E(-1, \dots, -1)$  is not regular if and only if one of the following conditions is satisfied (if  $r \geq m + n$  we can conclude this without using Le Potier vanishing theorem):

- i  $H^d(E(-1, \dots, -1) \otimes \mathcal{O}(-n_1, \dots, -n_s)) \neq 0$ ,
- ii there are  $s$  numbers  $h_1, \dots, h_s$  (where for any  $j = 1, \dots, s$ ,  $h_j = 0$  or  $h_j = n_j$  and  $0 < k_1, \dots, k_s < d$ ) such that  $H^{h_1 + \dots + h_s}(E(-1, \dots, -1) \otimes \mathcal{O}(-h_1, \dots, -h_s)) \neq 0$ .
- iii there are  $s$  numbers  $h_1, \dots, h_s$  (where for any  $j = 1, \dots, s$ ,  $-n_j \leq -h_j \leq 0$  at least one  $h_j = n_j$  and at least one  $h_j \neq n_j, 0$ ) such that  $H^{h_1 + \dots + h_s}(E(-1, \dots, -1) \otimes \mathcal{O}(-h_1, \dots, -h_s)) \neq 0$ .

The proof of Theorem 4.7 shows that the conditions [i] and [ii] give us direct summands  $\mathcal{O}(l_1, \dots, l_s)$  (where for any  $j = 1, \dots, s$   $l_j = 1$  or  $l_j = 0$  but  $(l_1, \dots, l_s) \neq (1, \dots, 1)$ ).

Let us consider the others conditions:

[iii] Up to a permutation of the factors of the multiprojective space we may assume that there is an integer  $l$  with  $1 \leq l < s$  and  $d - l$  integers  $a_{l+1}, \dots, a_s$  (where for any  $j = l + 1, \dots, s$   $-n_j \leq -a_j \leq 0$  and  $(a_{l+1}, \dots, a_s) \neq (0, \dots, 0)$ ) such that  $H^{n_1 + \dots + n_l + a_{l+1} + \dots + a_s}(E(-1, \dots, -1) \otimes \mathcal{O}(-n_1, \dots, -n_l, -a_{l+1}, \dots, -a_s)) \neq 0$ .

Let us consider the following exact sequences tensored by  $E$ :

$$\begin{aligned} 0 \rightarrow \mathcal{O}(-n_1 - 1, \dots, -n_l - 1, -a_{l+1} - 1, \dots, -a_s - 1) \rightarrow \dots \\ \dots \rightarrow \mathcal{O}(0, \dots, 0, -a_{l+1} - 1, \dots, -a_s - 1) \rightarrow 0, \end{aligned}$$

and the dual of

$$\begin{aligned}
& 0 \rightarrow \mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_s + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow \dots \\
& \dots \rightarrow \mathcal{O}(0, \dots, 0, a_{l+1} + 1, \dots, +a_{s-1} + 1) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow 0, \\
& \dots \\
& 0 \rightarrow \mathcal{O}(0, \dots, 0, a_{l+1} + 1, \dots, +a_{s-1} + 1) \boxtimes \Omega_{\mathbf{P}^{n_{s-1}}}^{a_{s-1}}(a_{s-1} + 1) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow \dots \\
& \dots \rightarrow \mathcal{O}(0, \dots, 0, a_{l+1} + 1, \dots, +a_{s-1} + 1) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow 0, \\
& 0 \rightarrow \mathcal{O}(0, \dots, 0, a_{l+1} + 1, \dots, +a_{s-1} + 1) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow \dots \\
& \dots \rightarrow \mathcal{O}(0, \dots, 0, a_{l+1} + 1, \dots, a_s + 1) \rightarrow 0,
\end{aligned}$$

By using the vanishing conditions in (1) we can conclude that

$$H^0(E \otimes (\mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_s + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1))^\vee) \neq 0.$$

On the other hand

$$\begin{aligned}
& H^{n_1 + \dots + n_l + a_{l+1} + \dots + a_s}(E \otimes \mathcal{O}(-n_1 - 1, \dots, -n_l - 1, -a_{l+1} - 1, \dots, -a_s - 1)) \cong \\
& \cong H^{n_{l+1} - a_{l+1} + \dots + n_s - a_s}(E^\vee \otimes \mathcal{O}(0, \dots, 0, -n_{l+1} - a_{l+1}, \dots, -n_s - a_s)).
\end{aligned}$$

Let us consider the following exact sequences tensored by  $E^\vee$ :

$$\begin{aligned}
& 0 \rightarrow \mathcal{O}(0, \dots, 0, -n_{l+1} - a_{l+1}, \dots, -n_s - a_s) \rightarrow \dots \\
& \dots \rightarrow \mathcal{O}(0, \dots, 0, -n_{l+1} - a_{l+1}, \dots, -n_{s-1} - a_{s-1}) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow 0, \\
& 0 \rightarrow \mathcal{O}(0, \dots, 0, -n_{l+1} - a_{l+1}, \dots, -n_{s-1} - a_{s-1}) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow \dots \\
& \dots \rightarrow \mathcal{O}(0, \dots, 0, -n_{l+1} - a_{l+1}, \dots, -n_{s-2} - a_{s-2}) \boxtimes \Omega_{\mathbf{P}^{n_{s-1}}}^{a_{s-1}}(a_{s-1} + 1) \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow 0, \\
& \dots \\
& 0 \rightarrow \mathcal{O}(0, \dots, 0, -n_{l+1} - a_{l+1}) \boxtimes \Omega_{\mathbf{P}^{n_{l+2}}}^{a_{l+2}}(a_{l+2} + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow \dots \\
& \dots \rightarrow \mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_{l+1} + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1) \rightarrow 0,
\end{aligned}$$

By using the vanishing conditions in (1) we can conclude that

$$H^0(E^\vee \otimes \mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_{l+1} + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1)) \neq 0.$$

So by arguing as in Theorem 1.2 we have that  $\mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_{l+1} + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1)$  is a direct summand of  $E$ .

(2)  $\Rightarrow$  (1). We prove it by induction on  $s$ .

For  $s = 2$   $\mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_{l+1} + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1)$  satisfies all the conditions of (1) By Theorem 3.13. Let us prove the inductive step from  $s - 1$  to  $s$ :

Let  $G \cong \mathcal{O}(0, \dots, 0) \boxtimes \Omega_{\mathbf{P}^{n_{l+1}}}^{a_{l+1}}(a_{l+1} + 1) \boxtimes \dots \boxtimes \Omega_{\mathbf{P}^{n_{s-1}}}^{a_{s-1}}(a_{s-1} + 1)$ .

We want to show that  $G \boxtimes \mathcal{O}$  and  $G \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1)$  satisfies all the conditions of (1):  
for any  $i = 1, \dots, \min(r, d) - 1$  and for any integer  $t$ ,

$$H^i(G(t+k_1, \dots, t+k_{s-1}) \boxtimes \mathcal{O}(t+k_s)) \cong \oplus_{p+q=i} H^p(G(t+k_1, \dots, t+k_{s-1})) \otimes H^q(\mathcal{O}(t+k_s)) = 0$$

whenever  $k_1 + \dots + k_s \geq -i$  and  $-n_j < k_j \leq 0$  for any  $j = 1, \dots, s$ .

In fact by the inductive hypothesis  $H^p(G(t+k_1, \dots, t+k_{s-1}))$  must be zero.

In the same way we can prove that  $G \boxtimes \Omega_{\mathbf{P}^{n_s}}^{a_s}(a_s + 1)$  satisfies all the conditions of (1).  $\square$

We finally specialize on rank two bundles giving the following statement:

**Proposition 4.9.** *Let  $n_1, \dots, n_s > 2$  and  $d = n_1 + \dots + n_s$ . Let  $E$  be a rank 2 vector bundle on  $X = \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_s}$  with  $\text{Reg}(E) = 0$ .*

*Then the following conditions are equivalent:*

1.  $H^1(E(-k_1, \dots, -k_s))$  whenever  $k_j \geq 0$  for any  $j = 1, \dots, s$  and  $k_1 + \dots + k_s \leq 1$
2.  $E \cong \mathcal{O}(l_1, \dots, l_s) \oplus \mathcal{O}(a_1, \dots, a_s)$  (where for any  $j = 1, \dots, s$   $l_j = 1$  or  $l_j = 0$  but  $(l_1, \dots, l_s) \neq (1, \dots, 1)$  and  $a_1, \dots, a_s \geq 0$ ).

*Proof.* Since  $\text{Reg}(E) = 0$ ,  $E$  is regular but  $E(-1, \dots, -1)$  not.

By the above proof we have that  $\mathcal{O}(l_1, \dots, l_s) \oplus \mathcal{O}(a_1, \dots, a_s)$  (where for any  $j = 1, \dots, s$   $l_j = 1$  or  $l_j = 0$  but  $(l_1, \dots, l_s) \neq (1, \dots, 1)$ ) is a direct summand of  $E$ .

The other summand must be  $\mathcal{O}(a_1, \dots, a_s)$  where  $a_1, \dots, a_s \geq 0$ .  $\square$

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